Commun. Math. Stat. (2015) 3:353–364
DOI 10.1007/s40304-015-0064-z

On the Linear Independence and Partition of Unity of Arbitrary Degree Analysis-Suitable T-splines

Jingjing Zhang[1] · **Xin Li**[1]

Received: 26 January 2015 / Revised: 27 May 2015 / Accepted: 8 June 2015 / Published online: 30 August 2015

Abstract Analysis-suitable T-splines are a topological-restricted subset of T-splines, which are optimized to meet the needs both for design and analysis (Li and Scott Models Methods Appl Sci 24:1141–1164, 2014; Li et al. Comput Aided Geom Design 29:63–76, 2012; Scott et al. Comput Methods Appl Mech Eng 213–216, 2012). The paper independently derives a class of bi-degree (d_1, d_2) T-splines for which no perpendicular T-junction extensions intersect, and provides a new proof for the linearly independence of the blending functions. We also prove that the sum of the basis functions is one for an analysis-suitable T-spline if the T-mesh is admissible based on a recursive relation.

Keywords T-splines · Analysis-suitable T-splines · Linear independence · Partition of unity · Isogeometric analysis

Mathematics Subject Classification 65D07

1 Introduction

T-splines were originally introduced as an alternative free-form geometric shape technology to solve many inherent limitations of standard NURBS representation in the industry [12,14]. Two main advantages of T-splines are local refinement [11,12] and watertightness [13]. Multiple NURBS patches can be merged into a single T-spline [7,14] and any trimmed NURBS model can be approximated with a watertight T-spline model [13] under any given tolerance. Thus, T-splines have emerged as an

✉ Xin Li
lixustc@ustc.edu.cn

1 School of Mathematical Science, USTC, Hefei, Anhui, China

important technology across several disciplines including industrial, architectural, and engineering design, manufacturing and engineering iso-geometric analysis.

The isogeometric analysis (for short, IGA) paradigm uses the smooth geometric basis as the basis for analysis, which is introduced in [6] and described in detail in [3]. With IGA, traditional design-through-analysis procedures such as geometry clean-up, defeaturing and mesh generation are simplified or eliminated entirely. Most of the early developments in isogeometric analysis focused on the behavior of NURBS basis functions [3,6] and later on T-splines [1], Hierarchical B-splines [17], PHT [10, 22] and LR B-splines [4]. In 2009, [2] discovered an example of a T-spline with linearly dependent blending functions, which means that the whole class of T-splines are not suitable for IGA. Thus, analysis-suitable T-splines (for short, AS T-splines), a class of T-splines which associated T-meshes have no intersections of T-junction extensions, were developed in [9,11] to meet the basic needs for IGA. The members of the class of T-splines are NURBS compatible, watertight, convex hull, affine invariant, always linear independent for any knot intervals, optimized local refinement [11] and characterized in terms of piecewise polynomial [8].

Linear independence and partition of unity for T-spline blending functions are two fundamental theoretical problems associated with T-splines. There are two different approaches to analysis the linear independence of T-spline blending functions by computing the nullity of the transform matrix [9,18,21] or dual basis [15,16]. The partition of unity for T-splines blending functions has not been well understood till now [19,20]. The present paper identifies a class of T-splines whose blending functions are guaranteed to be linearly independent using a different approach from [16] by computing the nullity of the transform matrix. Compared with the result in [9], the main contribution includes,

- we generalize the bi-cubic analysis-suitable T-splines [9] to any bi-degrees;
- we derive a recursive relation for AS T-splines and use the relation to prove the partition of unity property for AS T-splines.

The following paper is structured as follows. Pertinent background on T-splines is reviewed in Sect. 2. Section 3 proves that any analysis-suitable T-spline has linearly independent blending functions. Section 4 proves that the sum of the basis functions for an admissible analysis-suitable T-spline is one. The last section is conclusion and future work.

2 T-Splines

In the section, we prepare some basic notations and preliminary results for arbitrary degree T-splines [1,5].

Similar as the approach of [1], we define a T-spline based on the T-mesh in the index domain which is referred as an index T-mesh in the paper. A T-mesh is an important object to determine T-spline blending functions and how they are arranged with respect to each other. A T-mesh T for bi-degree (d_1, d_2) T-spline is a connection of all the elements of a rectangular partition of the index domain $[0, c+d_1] \times [0, r+d_2]$, where all rectangle corners (or vertices) have integer coordinates. Each vertex has a unique pair of index coordinate (δ_i, τ_i). An edge is a line segment connecting two vertices

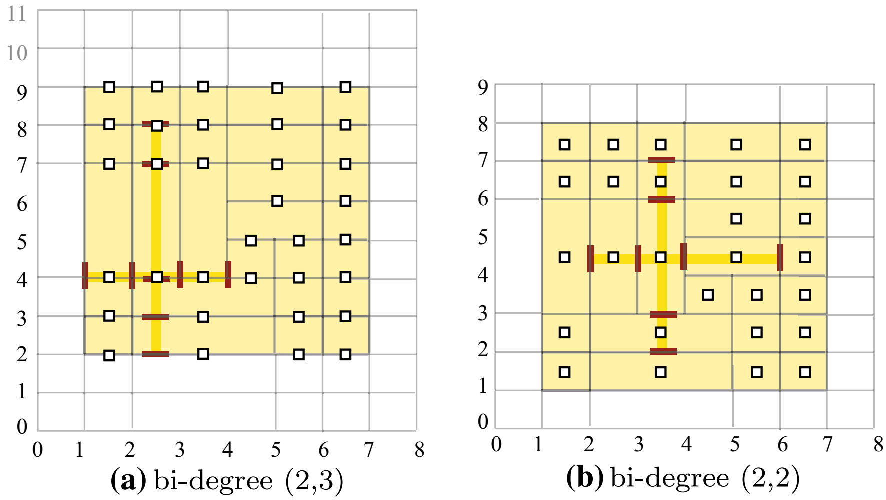


Fig. 1 The anchors and the local index vector for one blending function

in the T-mesh and no other vertices lying in the interior. And a face is a rectangle where no other edges and vertices in the interior. The valence of a vertex is the number of edges such that the vertex is an endpoint. For the interior vertices, we only allow valence three (called T-junctions) or four vertices. We adopt the notations $\vdash$, $\dashv$, $\perp$ and $\top$ to indicate the four possible orientations for the T-junctions. Denote the active region as rectangle region $[p, c + d_1 - p] \times [q, r + d_2 - q]$, here p and q are the maximal integers equal or less than $\frac{d_1+1}{2}$ and $\frac{d_2+1}{2}$, respectively. As we will see below, the active region carries the anchors that will be associated with the blending functions while the other indices will be needed for the definition of the blending function when the anchors are close to the boundary.

An anchor is a point in the index T-mesh which corresponds one blending function. If both d_1 and d_2 are odd, an anchor corresponds a vertex in the active region of the T-mesh, if both d_1 and d_2 are even, then an anchor corresponds the barycenter of a face in the active region of the T-mesh. And the index coordinate for the anchor is the index coordinate of the left-bottom vertex of the associated face. If d_1 is even and d_2 is odd or d_1 is odd and d_2 is even, an anchor is the middle point of a horizontal edge or a vertical edge in the active region of the T-mesh. The index coordinate for the anchor is the index coordinate of the left or bottom vertex of the associated edge.

For the i-th anchor $\mathbf{A}_i$, we define a local index vector $\boldsymbol{\delta}_\mathbf{i} \times \boldsymbol{\tau}_\mathbf{i}$ which is used to define the blending function $T_i(s, t)$. The values of $\boldsymbol{\delta}_\mathbf{i} = [\delta_i^0, \ldots, \delta_i^{d_1+1}]$ and $\boldsymbol{\tau}_\mathbf{i} = [\tau_i^0, \ldots, \tau_i^{d_2+1}]$ are determined as follows. From the i-th anchor in the T-mesh, we shoot a ray in the s and t direction traversing the T-mesh and collect a total of $d_1 + 2$ and $d_2 + 2$ knot indices to form $\boldsymbol{\delta}_\mathbf{i}$ and $\boldsymbol{\tau}_\mathbf{i}$, as shown in Fig. 1.

The indices correspond two global knot vectors $\mathbf{s} = [s_0, s_1, \ldots, s_{c+d_1}]$ and $\mathbf{t} = [t_0, t_1, \ldots, t_{r+d_2}]$. The end condition knots for $\mathbf{s}$ and $\mathbf{t}$ may have multiplicity $d_1 + 1$ and $d_2 + 1$; all the other knots are of multiplicity $\le d_1$ and $\le d_2$, respectively. Each edge is assigned with a knot interval which is the associated parametric length of the edge. The valid rules for the knot configuration require that the sums of the knot intervals on opposite sides of a face must be equal [14]. Then we are ready to define the blending function $T_i(s, t)$ associated with the i-th anchor, which is a tensor-product of degree d_1 and d_2 B-spline functions. The knot vectors are defined by the local knot vectors $\boldsymbol{\delta}_\mathbf{i} \times \boldsymbol{\tau}_\mathbf{i}$,

$$T_i(s, t) = B[\mathbf{s}_i](s)B[\mathbf{t}_i](t), \tag{2.1}$$

where

$$\mathbf{s}_i = [s_{\sigma_i^0}, s_{\sigma_i^1}, \ldots, s_{\sigma_i^{d_1+1}}] \quad \text{and} \quad \mathbf{t}_i = [t_{\tau_i^0}, t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2+1}}] \tag{2.2}$$

are subsequences of **s** and **t**, respectively.

A T-spline space is finally given as the span of all these blending functions and a T-spline surface is defined as

$$\mathbf{C}(s, t) = \sum_{i=1}^{n_A} \mathbf{C}_i T_i(s, t), \tag{2.3}$$

where $\mathbf{C}_i = (\omega_i x_i, \omega_i y_i, \omega_i z_i, \omega_i) \in \mathbb{P}^3$ are homogeneous control points, $\omega_i \in \mathbb{R}$ are weights, $T_i(s, t)$ are blending functions, and n_A is the number of control points or anchors.

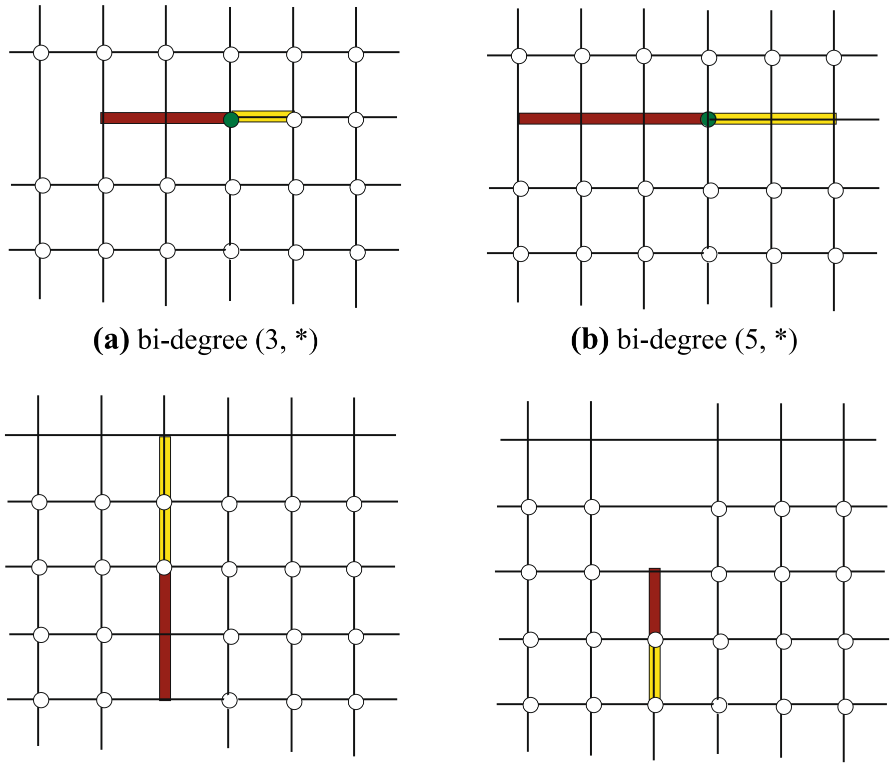
Analysis-suitable T-splines are defined in terms of T-junction extensions. For example, the extension for a T-junction of type $\vdash$ is a line segment $[\underline{i}, \bar{i}] \times \{\tau_i\}$. $\underline{i}$ and $\bar{i}$ are determined such that the edges $[\underline{i}, \delta_i) \times \{\tau_i\}$ have $[\frac{d_1+1}{2}]$ intersections with the T-mesh and the edges $(\delta_i, \bar{i}] \times \{\tau_i\}$ have $[\frac{d_1}{2}]$ intersections with the T-mesh. Here $[d]$ means the maximal integer less or equal d. For a T-junction of type $\dashv$, we can similarly define the extension except the number of intersections that are exchanged. Also, we can define the extensions for the other kinds of T-junctions $\perp$, $\top$, where uses degree d_2 instead of d_1. All these extension examples are illustrated in Fig. 2.

Definition 2.1 For a bi-degree (d_1, d_2) T-spline, a T-mesh is called analysis-suitable (for short, AS T-mesh) if the extensions for all the T-junctions $\vdash$ and $\dashv$ do not intersect the extensions for all the T-junctions $\perp$ and $\top$. A T-spline defined on an analysis-suitable T-mesh is called an analysis-suitable T-spline, for short AS T-spline.

AS T-meshes have two key properties (Lemmas 2.2 and 2.3) which will be used in the following section. As these two lemmas have also been proved in [16] (Lemma 3.2 (a) and (b)), so we omit the proof here.

Lemma 2.2 *In an analysis-suitable T-mesh* $\mathbb{T}$, *for any anchor* $\mathbf{A}_i$, *let* $TF(\mathbf{A}_i)$ *be the union of all the rectangles* $R_i^{j,k} \doteq (\sigma_i^j, \sigma_i^{j+1}) \times (\tau_i^k, \tau_i^{k+1})$. *Here* $j = 0, \ldots, d_1$, $k = 0, \ldots, d_2$, *then there are no T-mesh vertices inside* $TF(\mathbf{A}_i)$.

Lemma 2.3 *In an analysis-suitable T-mesh* $\mathbb{T}$, *for any anchor* $\mathbf{A}_i$, *let* $hSK(\mathbf{A}_i)$ *be the union of all the edge segments* $[\sigma_i^0, \sigma_i^{d_1+1}] \times \{\tau_i^j\}$, $j = 0, 1, \ldots, d_2+1$ *and* $vSK(\mathbf{A}_i)$ *be the union of all the edge segments* $\{\sigma_i^j\} \times [\tau_i^0, \tau_i^{d_2+1}]$, $j = 0, 1, \ldots, d_1 + 1$. *Then*



(a) bi-degree (3, *) **(b)** bi-degree (5, *) **(c)** bi-degree (*, 4) **(d)** bi-degree (*, 2)

Fig. 2 The extensions for four different kinds of T-junctions

$hSK(\mathbf{A}_i)$ lies on the edges of T-mesh $\mathbb{T}$ or lies on the extensions of all T-junctions $\vdash$ and $\dashv$ on $hSK(\mathbf{A}_i)$, and $vSK(\mathbf{A}_i)$ lies on the edges of T-mesh $\mathbb{T}$ or on the extensions of T-junctions $\perp$ and $\top$ on $vSK(\mathbf{A}_i)$.

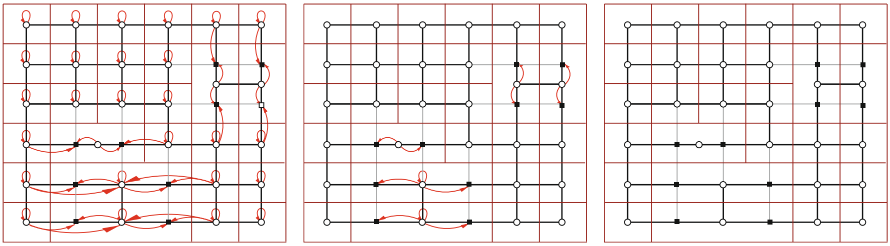
Definition 2.4 A T-mesh is admissible, if the vertex (i, j) is not $\perp$ or $\top$ when $0 \leq i \leq d_1$ or $c \leq i \leq c + d_1$, and is not $\vdash$ or $\dashv$ when $0 \leq j \leq d_2$ or $r \leq j \leq r + d_2$.

3 Linear Independence

In this section, we generalize the method in [9] to prove that any bi-degree (d_1, d_2) analysis-suitable T-splines have linear independent blending functions.

3.1 NURBS Conversion

Each T-spline can be converted into the underlying NURBS form $\mathcal{N}(s,t) = \sum_{k=1}^{n_P} \mathcal{N}_k \quad N_k(s,t)$, where n_P is the number of NURBS anchors. And $\mathcal{N}_i = \sum_{j=1}^{n_A} m_{i,j}\mathcal{C}_j$. This relationship can be written in a matrix form $M\mathbb{T} = \mathbb{P}$, where M is called the T-spline-to-NURBS transform matrix. If all the elements of row j of



(a) Influence graph. **(b)** First pruning. **(c)** Fully pruned.

Fig. 3 A T-mesh and its influence graph

M are zero except m_{ji}, column i is called an innocuous column. Column reduction is the operation of removing an innocuous column from M along with any zero rows that the column removal may have introduced. It is evident that column reduction operator will preserve the nullity of the matrix.

We can visualize column reduction by using a directed graph G drawn on a T-mesh that we call it an influence graph. G contains two types of anchors: T-anchors corresponding to the anchors for a T-spline, and N-anchors corresponding to the anchors for the underlying tensor-product NURBS by extending all the T-junctions to the boundary. If m_{ij} is non-zero, an edge is drawn from the j-th T-anchor to the i-th N-anchor. The valence of an N-anchor is the number of edges that point to it. The valence of a T-anchor is the number of edges originating from it. An innocuous anchor is any T-anchor that points to at least one N-anchor of valence one. The operation of pruning a graph is the graphical equivalent of column reduction, and consists of eliminating an innocuous anchor, edges originating from it, and any N-anchors that no longer are pointed to. A graph from which all innocuous anchors have been pruned is said to be fully pruned. A subgraph of G consists of any set of T-anchors, all N-anchors pointed to by those T-anchors, and all edges connecting those anchors. A V2-subgraph is a subgraph whose T-anchors all have a valence of at least two. A fully pruned graph is either empty, or consists of one or more V2-subgraphs.

Lemma 3.1 *If the fully pruned influence graph for a T-mesh has no V2-subgraphs, the T-spline has linearly independent blending functions.*

Proof See [9] for details. □

Figure 3 shows the pruning process of a bi-quadratic T-spline defined on the T-mesh in Fig. 1.

3.2 Linear Independence

This section presents that an AS T-mesh assures linear independence of the T-spline's blending functions.

Suppose $\widehat{\mathbf{s}}$ is a subsequence of $\mathbf{s}$, then the associated B-spline for knot vector $\widehat{\mathbf{s}}$ is a linear combination of all the B-splines for the global knot vectors $\mathbf{s}$, i.e., $B[\widehat{\mathbf{s}}](s) =$

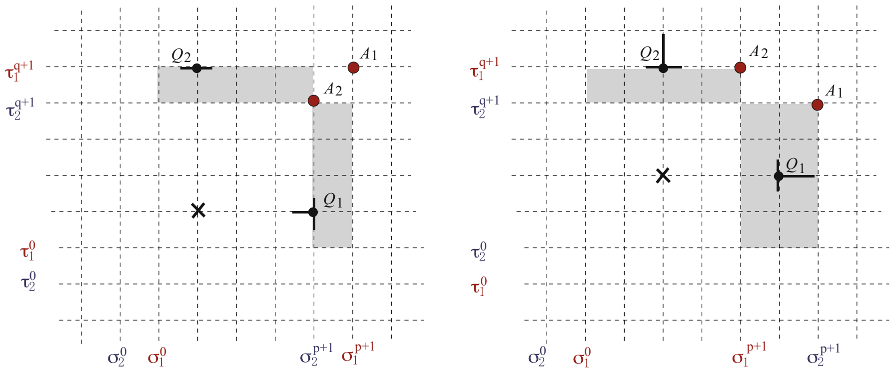


Fig. 4 Illustration for the proof

$\sum_{j=1}^{t} d_j B_j(s)$, where the d_j results from the knot insertions. We define $\mathbb{F}(\widehat{\mathbf{s}}) = \{j|d_j \neq 0\}$ which are the indices of all contributed B-splines and define $L(\widehat{\mathbf{s}})$ the biggest index in $\mathbb{F}(\widehat{\mathbf{s}})$. Similarly, we can define $\mathbb{F}(\widehat{\mathbf{t}})$ and $L(\widehat{\mathbf{t}})$. The footprint $\mathbb{F}(\mathbf{A}_i)$ for a T-anchor $\mathbf{A}_i$ is defined as $\mathbb{F}(\mathbf{A}_i) = \mathbb{F}(\mathbf{s}_i) \times \mathbb{F}(\mathbf{t}_i)$, which are all the indices of N-anchors pointed by the T-anchor and let $LL_i = (L(\mathbf{s}_i), L(\mathbf{t}_i))$ (Fig. 4).

Lemma 3.2 *Suppose we are given two knot vectors* $\mathbf{s}_1 = (s_{i_0}, s_{i_1}, \ldots, s_{i_{d+1}})$ *and* $\mathbf{s}_2 = (s_{i_k}, s_{i_{k+1}}, \ldots, s_{i_{d+1+k}})$, $k > 0$, *which are both subsequences of global knot vector, then* $L(\mathbf{s}_1) < L(\mathbf{s}_2)$.

Proof It is evident that $L(\mathbf{s}_1) \leq L(\mathbf{s}_2)$ and if $s_{i_0} < s_{i_k}$, $L(\mathbf{s}_1) < L(\mathbf{s}_2)$. Otherwise, suppose $s_{i_0} = s_{i_k}$, and $s_{i_0} = \cdots = s_{i_{m_1}} \neq s_{i_{m_1+1}}$ and $s_{i_k} = \cdots = s_{i_{k+m_2}} \neq s_{i_{k+m_2+1}}$, then $m_1 = m_2 + k$. According to the definition of $L(\mathbf{s}_1)$ and $L(\mathbf{s}_2)$, the multiplicity of $s_{L(\mathbf{s}_1)}$ and $s_{L(\mathbf{s}_2)}$ should be m_1 and m_2, respectively, i.e., $s_{L(\mathbf{s}_1)} = \cdots = s_{L(\mathbf{s}_1)+m_1} \neq s_{L(\mathbf{s}_1)+m_1+1}$, and $s_{L(\mathbf{s}_2)} = \cdots = s_{L(\mathbf{s}_2)+m_2} \neq s_{L(\mathbf{s}_2)+m_2+1}$. As $m_1 > m_2$, so $L(\mathbf{s}_1) < L(\mathbf{s}_2)$. □

Theorem 3.3 *The blending functions for an analysis-suitable T-spline are linearly independent.*

Proof If the fully pruned influence graph for the T-mesh has no V2-subgraphs, then the theorem can be directly derived from Lemma 3.1. Otherwise, suppose it has a V2-subgraph $\tilde{G}$, denote anchor $\mathbb{LL}$ to be the bottommost N-anchor in the V2-subgraph (if there are more than one N-anchor, we choose the leftmost one). Thus, there must exist at least two T-anchors, denoted as $\mathbf{A}_1, \mathbf{A}_2$ such that $LL_1 = LL_2 = \mathbb{LL}$. Suppose the index coordinates for $\mathbf{A}_1$ and $\mathbf{A}_2$ are (σ_1, τ_1) and (σ_2, τ_2), respectively. If $\sigma_1 = \sigma_2$ or $\tau_1 = \tau_2$, then $LL_1 \neq LL_2$ by Lemma 3.2. Otherwise, we have the following two cases. In both cases, we denote $\delta = \max(\sigma_1^0, \sigma_2^0)$ and $\tau = \max(\tau_1^0, \tau_2^0)$.

1. If $\sigma_1 > \sigma_2$ and $\tau_1 > \tau_2$ (or, if $\sigma_1 < \sigma_2$ and $\tau_1 < \tau_2$):
   Since $L(\mathbf{t}_1) = L(\mathbf{t}_2)$, according to Lemma 3.2, the common indices for $\mathbf{t}_1$ and $\mathbf{t}_2$ between $[\tau, \tau_2^{q+1}]$ cannot match exactly. Thus, there exists T-junction $Q_1 = (\delta_1, \tau_1)$, $\vdash$ or $\dashv$, in rectangle $[\sigma_2^{p+1}, \sigma_1^{p+1}] \times [\tau, \tau_2^{q+1}]$. And if Q_1 lies on the

index line σ_2^{p+1}, it can only be $\dashv$. Thus, the extension of Q_1 covers all edges $[\delta, \sigma_2^{p+1}] \times \{\tau_1\}$ following Lemma 2.3. With the same analysis for $\mathbf{s}_1$ and $\mathbf{s}_2$, we can conclude that there exists a T-junction $Q_2 = (\delta_2, \tau_2)$, $\perp$ or $\top$ which extension covers all edges $\{\delta_2\} \times [\delta, \sigma_2^{p+1}]$. As $\delta_2 \in [\delta, \sigma_2^{p+1}]$ and $\tau_1 \in [\tau, \tau_2^{q+1}]$, the two extensions intersect.

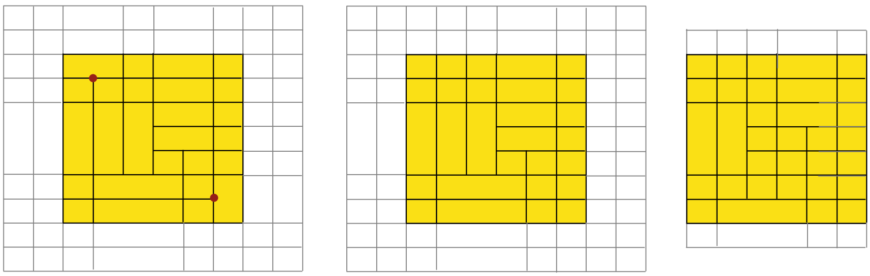
2. If $\sigma_1 > \sigma_2$ and $\tau_1 < \tau_2$ (or, if $\sigma_1 < \sigma_2$ and $\tau_1 > \tau_2$):
   As $L(\mathbf{t}_1) = L(\mathbf{t}_2)$, according to Lemma 3.2, the common indices for $\mathbf{t}_1$ and $\mathbf{t}_2$ between $[\tau, \tau_1^{q+1}]$ cannot match exactly. Thus, there exists T-junction $Q_1 = (\delta_1, \tau_1)$, $\vdash$ or $\dashv$, in rectangle $[\sigma_1^{p+1}, \delta] \times [\tau, \tau_1^{q+1}]$. And if Q_1 lies on the index line σ_1^{p+1}, it can only be $\dashv$. Thus, the extension of Q_1 covers all edges $[\delta, \sigma_2^{p+1}] \times \{\tau_1\}$. With the same analysis for $\mathbf{s}_1$ and $\mathbf{s}_2$, we can conclude that there exists a T-junction $Q_2 = (\delta_2, \tau_2)$, $\perp$ or $\top$ which extension covers all edges $\{\delta_2\} \times [\tau, \tau_1^{q+1}]$. As $\delta_2 \in [\delta, \sigma_2^{p+1}]$ and $\tau_1 \in [\tau, \tau_1^{q+1}]$, the two extensions intersect.

Thus, all cases assure that AS T-splines have no V2-subgraphs and our proof follows from Lemma 3.1. □

4 Partition of Unity

In this section, we further prove that the sum of the blending functions for an AS T-spline is one if the T-mesh is admissible. The basic idea is based on the following recursive relation for AS T-splines.

Given a bi-degree (d_1, d_2) AS T-spline defined on T-mesh $\mathbb{T}$, a new T-mesh $\mathbb{T}_{\alpha,\beta}^{d_1,d_2}$ is defined according to the following rules. Let α_1 and β_1 be the maximal integer equal or less than $\frac{\alpha+1}{2}$ and $\frac{\beta+1}{2}$, respectively. First, we extend each T-junction $\perp$ and $\top$ α_1 bays and extend each T-junction $\vdash$ and $\dashv$ β_1 bays to create a new T-mesh. Here a bay means the index intervals for two intersections of the extension in the T-mesh. And then we create T-mesh $\mathbb{T}_{\alpha,\beta}^{d_1,d_2}$ from the T-mesh which lies in the rectangle region $[\alpha, c + d_1 - \alpha] \times [\beta, r + d_2 - \beta]$. Fig. 5 illustrates an admissible T-mesh $\mathbb{T}_{0,0}^{3,3}$ and the corresponding T-mesh $\mathbb{T}_{2,1}^{3,3}$.



(a) Not a bi-cubic admissible T-mesh **(b)** Corresponding admissible T-mesh $\mathbb{T}_{0,0}^{3,3}$ **(c)** admissible T-mesh $\mathbb{T}_{2,1}^{3,3}$

Fig. 5 Admissible T-mesh $\mathbb{T}_{\alpha,\beta}^{d_1,d_2}$

Lemma 4.1 *If T-mesh* $\mathbb{T}_{0,0}^{d_1,d_2}$ *is an admissible AS T-mesh, then for all* $0 \leq \alpha \leq d_1$ *and* $0 \leq \beta \leq d_2$, $\mathbb{T}_{\alpha,\beta}^{d_1,d_2}$ *is also an admissible AS T-mesh.*

Proof This can be derived from the definition of admissible T-mesh directly. □

Lemma 4.2 *Suppose the blending functions for a bi-degree* (d_1, d_2) *AS T-spline defined on T-mesh* $\mathbb{T}_{0,0}^{d_1,d_2}$ *are* $T_k^{d_1,d_2}(s,t)$, $k = 1, \ldots, n_1$ *and those for a bi-degree* $(d_1 - \alpha, d_2 - \beta)$ *AS T-spline defined on T-mesh* $\mathbb{T}_{\alpha,\beta}^{d_1,d_2}$ *are* $T_k^{d_1-\alpha,d_2-\beta}(s,t)$, $k = 1, \ldots, n_2$, *then*

$$\sum_{k=1}^{n_1} T_k^{d_1,d_2}(s,t) = \sum_{k=1}^{n_2} T_k^{d_1-\alpha,d_2-\beta}(s,t). \tag{4.1}$$

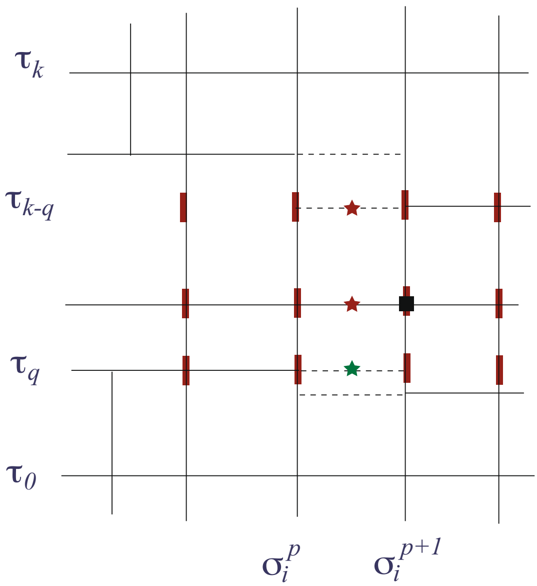
Proof We only need to prove the lemma for (α, β) being $(1, 0)$ and $(0, 1)$ because the other cases can be proved recursively. And for symmetry, the only case we need to prove is $\alpha = 1, \beta = 0$. Also in order to make the notations to be simple, we can assume that d_1 and d_2 are both odd since the other cases are exactly similar.

The main proof has two steps. The first step is to prove that for any blending function $T_i(s,t) = B[\mathbf{s}_i](s) \times B[\mathbf{t}_i](t)$, where $\mathbf{s}_i = [s_{\sigma_i^0}, s_{\sigma_i^1}, \ldots, s_{\sigma_i^{2p+2}}]$ and $\mathbf{t}_i = [t_{\tau_i^0}, t_{\tau_i^1}, \ldots, t_{\tau_i^{2q+2}}]$, for a vertex $\mathbf{C}_i^1$ defined on T-mesh $\mathbb{T}_{0,0}^{d_1,d_2}$ can be expressed into the combination of bi-degree $(2p, 2q+1)$ blending functions defined on $\mathbb{T}_{1,0}^{d_1,d_2}$. It is obvious that $T_i(s,t)$ can be written into the combination of B-splines $B[\mathbf{s}_i^l](s) \times B[\mathbf{t}_i](t)$ and $B[\mathbf{s}_i^r](s) \times B[\mathbf{t}_i](t)$, here $\mathbf{s}_i^l = [s_{\sigma_i^0}, s_{\sigma_i^1}, \ldots, s_{\sigma_i^{2p+1}}](s)$ and $\mathbf{s}_i^r = [s_{\sigma_i^1}, s_{\sigma_i^2}, \ldots, s_{\sigma_i^{2p+2}}](s)$, respectively. So we should prove that T_i^l and T_i^r are the linear combinations of the bi-degree $(2p, 2q+1)$ blending functions in $\mathbb{T}_{1,0}^{d_1,d_2}$. We only consider B-spline T_i^l and T_i^r can be proved exactly in the same method.

Denote the two adjacent edges of $\mathbf{C}_i^1$ by $\mathbf{C}_i^{2l}$ and $\mathbf{C}_i^{2r}$. Referring to Fig. 6, in T-mesh $\mathbb{T}_{1,0}^{d_1,d_2}$, suppose that the indices of the T-nodes both on edges $\{\sigma_i^p\} \times [\tau_i^0, \tau_i^{2q+2}]$ and $\{\sigma_i^{p+1}\} \times [\tau_i^0, \tau_i^{2q+2}]$ are $\tau_0, \ldots, \tau_k, k \geq 2q+2$. Then it is sufficient to prove that the indices of the knot vectors for the blending functions which are associated with edges $[\sigma_i^p, \sigma_i^{p+1}] \times \{\tau_j\}$, $j = q, \ldots, k-q$ in the s-direction are all $[\sigma_i^0, \sigma_i^1, \ldots, \sigma_i^{2p+1}]$.

Suppose there is an edge $[\sigma_i^p, \sigma_i^{p+1}] \times \{\tau_n\}$, which indices of the knot vector for the blending function is not $[\sigma_i^0, \sigma_i^1, \ldots, \sigma_i^{2p+1}]$. It is obvious that $\tau_i^{q+1} \neq \tau_n$. So we can assume that $\tau_i^{q+1} > \tau_n$ since the arguments for the two cases $\tau_n > \tau_i^{q+1}$ and $\tau_i^{q+1} > \tau_n$ are similar. As the indices of the knot vector for the blending function is not $[\sigma_i^0, \sigma_i^1, \ldots, \sigma_i^{2p+1}]$, so there exists a T-junction $Q_1 = (\delta_1, \tau_1)$, being $\perp$ or $\top$, in rectangle $[\sigma_i^0, \sigma_i^{p+1}] \times [\tau_n, \tau_i^{q+1}]$. According to Lemma 2.3, the extension of Q_1 covers all edges $\{\delta_1\} \times [\tau_i^0, \tau_i^{q+1}]$. With the same analysis for t-direction, we can conclude that there exists a T-junction $Q_2 = (\delta_2, \tau_2)$, being $\vdash$ or $\dashv$, which extension covers all edges $[\sigma_i^0, \sigma_i^{p+1}] \times \{\tau_2\}$. As $\delta_2 \in [\sigma_i^0, \sigma_i^{p+1}]$ and $\tau_1 \in [\tau_i^0, \tau_i^{q+1}]$, so the two extensions intersect, which completes the proof of the first step.

Fig. 6 Insert a knot into t-direction to find presentation blending functions and points



The second step is to prove Eq. (4.1). The main idea is to compute the contribution of blending functions $T_k^{d_1,d_2}(s,t)$ to $T_k^{d_1-1,d_2}(s,t)$. For any blending function $T_k^{d_1-1,d_2}(s,t)$, with the knot vector in s-direction being $[s_{\sigma_k^0}, s_{\sigma_k^1}, \ldots, s_{\sigma_k^{2p+1}}]$, we first determine the indices k_i such that the contribution of $T_{k_i}^{d_1,d_2}(s,t)$ is not zero. According to the proof of first step, the indices can be divided into two parts, l_i and r_i. The knot vectors in s-direction for the blending functions of l_i are $[*, s_{\sigma_k^0}, s_{\sigma_k^1}, \ldots, s_{\sigma_k^{2p+1}}]$ and those for r_i are $[s_{\sigma_k^0}, s_{\sigma_k^1}, \ldots, s_{\sigma_k^{2p+1}}, *]$. Here $*$ means the index which could be different. Thus the sum of the contributions of all these l_i blending functions to $T_k^{d_1-1,d_2}(s,t)$ is $\frac{s-s_{\sigma_k^0}}{s_{\sigma_k^{2p+1}}-s_{\sigma_k^0}}$ and the sum of the contribution of all these r_i blending functions to $T_k^{d_1-1,d_2}(s,t)$ is $\frac{s_{\sigma_k^{2p+1}}-s}{s_{\sigma_k^{2p+1}}-s_{\sigma_k^0}}$. Thus, the sum of the contribution of all blending functions $T_i^{d_1,d_2}(s,t)$ to $T_k^{d_1-1,d_2}(s,t)$ is one, which is Eq. (4.1). □

Theorem 4.3 *The sum of the basis functions for an T-spline defined on an admissible AS T-mesh is one.*

Proof It is obvious that $\sum_{k=1}^{n_t} T_k^{0,0}(s,t) = 1$. Thus, the theorem can be directly derived from Lemmas 4.1 and 4.2. □

5 Conclusion

The paper generalizes bi-cubic AS T-splines to arbitrary degrees AS T-splines using a different approach from [16]. We also prove the sum of the blending functions for an admissible AS T-spline is one based on a recursive relation for AS T-splines, which can also be used to derive the hodograph formula for AS T-splines. As we can see from the present paper, The class of AS T-splines is a minor topology restricted to T-splines. And any T-spline can be represented as an AS T-spline defined on another T-mesh. Based on the dimension results in [8], we can also derive an optimized local refinement algorithm for any AS T-spline and characterize the AS T-splines spaces according to the linear independence. Future papers will focus on the degree elevation of AS T-splines and local degree elevation of AS T-splines.

Acknowledgments This work was supported by the NSF of China (No.11031007, No.60903148), the Chinese Universities Scientific Fund, SRF for ROCS SE, the CAS Startup Scientific Research Foundation and NBRPC 2011CB302400.

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